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$\kappa -$ FRÉCHET-URYSOHN PROPERTIES IN RECTIFIABLE SPACES

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Abstract: A topological space G is called a rectifiable space if there is a homeomorphism $\varphi: G\times G \to G\times G$ and an element $e\in G$ such that $\pi_1 \mathrm{o} \varphi = \pi_1$ and for every $x\in G$ we have $\varphi(x,x)=(x,e)$, where $\pi_1 : G\times G \to G$ is the projection to the first coordinate. Then, φ is called a rectification on G and e is a right unit element of G. Recently, rectifiable spaces have been studied by many authors who have put many open questions that have yet to be answered. In this article, we give κ-Fréchet-Urysohn properties in rectifiable spaces. These findings are used to generalize a result in [8].

Key words: Rectifiable space; κ-Fréchet-Urysohn space; strongly κ-Fréchet-Urysohn space; first-countable space; compact subset.

1. Introduction

In 1987, M. M. Choban introduced rectifiable spaces and give some of their properties ([1]). Since then, rectifiable spaces have been studied by many other authors (see [3, 4, 5]).

In this article, we give κ-Fréchet-Urysohn properties in rectifiable spaces:

(1) If every compact subset of a rectifiable space G is κ-Fréchet-Urysohn, then every compact subset of G is strongly κ-Fréchet-Urysohn.

(2) The product of a κ-Fréchet-Urysohn rectifiable space G with a first-countable space M is strongly κ-Fréchet-Urysohn.

With these results, we extend a result in [8].

Throughout this article, all spaces are T_1 and ¥ denotes the set of all natural numbers.

2.Theoretical bases and research methods

2.1. Theoretical bases

Lemma 1.1 ([1]). *A topological space G is rectifiable if and only if there exist $e\in G$ and two continuous maps $p: G\times G \to G$, $q: G\times G\to G$, such that for any $x\in G$, $y\in G$ the next identities hold:*

$$p\big(x, q(x,y)\big) = q\big(x, p(x,y)\big) = y$$

and $q(x,x) = e$.

Remark 1.2 ([1]). Let G be a rectifiable space and $x\in G$, we have

$$p(x,e) = p(x, q(x,x)) = x.$$

Moreover, we sometimes write xy instead of $p(x,y)$ for any $x, y\in G$ and AB instead of $p(A,B)$ for any $A, B\subset G$.

Lemma 1.3 ([4]). *Let G be a rectifiable space. Fixed a point $x\in G$, then $f_x, g_x : G\to G$ defined with $f_x(y) = p(x,y)$ and $g_x(y) = q(x,y)$, for each $y\in G$, are homeomorphism, respectively.*

Lemma 1.4 ([6]). *Let G be a rectifiable space, $A\subset G$ and U be an open set in G. Then, $p(A,U)$ and $q(A,U)$ are open subsets in G.*

Lemma 1.5 ([6]). *Let G be a rectifiable space and $x\in G$. Then, the following statements hold.*

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(1) *If U is an open neighborhood of x, then there exists an open neighborhood V of e in G such that* $xV \subset U$;

(2) *If U is an open neighborhood of e in G, then xU is an open neighborhood of x, and there exists an open neighborhood V of e in G such that* $q(xV, x) \subset U$.

Definition 1.6 ([7]). A space X is *κ-Fréchet-Urysohn at a point* $x \in X$ if for each open set U of X with $x \in \overline{U}$, there is a sequence $\{x_n : n \in \mathbb{N}\} \subset U$ converging to x. A space X is *κ-Fréchet-Urysohn* if it is κ-Fréchet-Urysohn at each point of X.

Definition 1.7 ([7]). A space X is *strongly κ-Fréchet-Urysohn at a point* $x \in X$ if for each decreasing open family $\{O_n : n \in \mathbb{N}\}$ of X with $x \in \bigcap_{n\in\mathbb{N}} \overline{O_n}$, there are $x_n \in O_n$, $n \in \mathbb{N}$, coverging to x. A space X is *strongly κ-Fréchet-Urysohn* if it is strongly κ-Fréchet-Urysohn at each point of X.

Remark 1.8 ([7]). Every strongly κ-Fréchet-Urysohn space is κ-Fréchet-Urysohn.

Lemma 1.9 ([8]). *If G is a κ-Fréchet-Urysohn rectifiable space, then it is strongly κ-Fréchet-Urysohn.*

Lemma 1.10 ([3]). *If G is a rectifiable space, then G is regular.*

Lemma 1.11 ([2]). *If U is open in X, then $\overline{U \cap A} = \overline{U \cap \overline{A}}$ for every $A \subset X$.*

2.2. Research methods

We used theoretical research methods in the course of conducting this study. We also carried out a literature review to pave the way for finding new results.

3. Results and evaluation

3.1. Results

Theorem 1.1. *If every compact subset of a rectifiable space G is κ-Fréchet-Urysohn, then every compact subset of G is strongly κ-Fréchet-Urysohn.*

Proof. Let A be a compact subset of G. Then A is closed and κ-Fréchet-Urysohn by the assumption and Lemma 1.10. Now, we prove that A is strongly κ-Fréchet-Urysohn. Indeed, suppose that $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of open subsets of A with $a \in \bigcap_{n\in\mathbb{N}} \overline{A_n}^A$. Then,

* *Case 1*. If $a \notin \overline{A \setminus \{a\}}^A$, then the set $\{a\}$ is open in A. Moreover, since $a \in \overline{A_n}^A$ for every $n \in \mathbb{N}$, it implies that $a \in A_n$ for every $n \in \mathbb{N}$. Hence, the sequence $\{a_n\}$ with $a_n = a \in A_n$ for every $n \in \mathbb{N}$ converges to a. Thus, A is strongly κ-Fréchet-Urysohn.

* *Case 2*. If $a \in \overline{A \setminus \{a\}}^A$, then since the set $A \setminus \{a\}$ is open in A and A is κ-Fréchet-Urysohn, there exists a sequence $\{a_n\} \subset A \setminus \{a\}$ converging to a. For each $n \in \mathbb{N}$, we put

$$B = q(a, A),\ B_n = q(a, A_n)$$

and $b_n = q(a, a_n)$.

Since A is closed in G and $B = q(a, A) = g_a(A)$ with g_a is a homeomorphism by Lemma 1.3, the set B is also closed in G. Moreover,

(1) We have $e = q(a,a) \in q\left(a, \overline{A_n}\right) \subset \overline{q(a, A_n)} = \overline{B_n} \subset \overline{B} = B$ by the continuity of the mapping $q(a,.)$, $b_n \in B \setminus \{e\}$ for each $n \in \mathbb{N}$ and the sequence $\{b_n\}$ converges to e.

(2) There is a sequence $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of e such that $b_n \notin p(V_n, V_n)$ for each $n \in \mathbb{N}$. Indeed, since G is T_1-space and $e \neq b_n$ for every $n \in \mathbb{N}$, there is an open neighborhood U_n of e such that $b_n \notin U_n$ for every $n \in \mathbb{N}$. Furthermore, since $p(e,e) = e$ and p is continuous, for each open neighborhood U_n of e, there exist two open

neighborhoods $W_n, W_n^{/}$ of e such that $p\left(W_n, W_n^{/}\right) \subset U_n$. Now, for each $n \in \mathbb{N}$, if we put $V_n = W_n \cap W'_n$, then V_n is an open neighborhood of e and $p(V_n, V_n) \subset U_n$. It implies that $b_n \notin p(V_n, V_n)$ for every $n \in \mathbb{N}$.

(3) Let $C_n = p(b_n, B_n \cap V_n)$ for every $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, $e \in \overline{B_n} \cap V_n \subset \overline{\overline{B_n} \cap V_n} = \overline{B_n \cap V_n}$ by Lemma 1.11. Moreover, for each $n \in \mathbb{N}$, we have $b_n \in \overline{C_n}$ and $e \notin \overline{C_n}$. Indeed, for every $n \in \mathbb{N}$, it follows from the continuity of the mapping $p(a,.)$ that

$$b_n = p(b_n, e) \in p\left(b_n, \overline{B_n \cap V_n}\right) \subset \overline{p(b_n, B_n \cap V_n)} = \overline{C_n}.$$

Next, since $b_n \notin p(V_n, V_n)$ for every $n \in \mathbb{N}$, we have

$$b_n V_n \cap p(V_n, V_n) V_n = \varnothing.$$

Moreover, since $V_n \subset p(V_n, V_n) V_n$ for every $n \in \mathbb{N}$, it implies that

$$b_n V_n \cap V_n = \varnothing.$$

On the other hand, since $V_n \cap C_n \subset V_n \cap b_n V_n$ for every $n \in \mathbb{N}$, we have

$$V_n \cap C_n = \varnothing.$$

Hence, $e \notin \overline{C_n}$ for every $n \in \mathbb{N}$.

Now, we put

$$D = \bigcup_{n \in \mathbb{N}} C_n \text{ and } S = \{e\} \cup \{b_n : n \in \mathbb{N}\}.$$

Then, $D \subset \bigcup_{n \in \mathbb{N}} p(b_n, B_n) \subset p(S, B) = SB.$

In the following we shall verify that the subset SB of G is closed and κ-Fréchet-Urysohn. Clearly, S is compact. Moreover, since A is compact and $B = q(a, A) = g_a(A)$ with g_a is a homeomorphism by Lemma 1.3, B is also compact. Therefore, $S \times B$ is compact. Furthermore, since p is continuous, $p(S, B) = SB$ is compact. Thus, SB is closed and κ-Fréchet-Urysohn by our assumption.

Since $\overline{D} = \overline{\overline{D}} = \overline{\bigcup_{n \in \mathbb{N}} C_n} = \overline{\bigcup_{n \in \mathbb{N}} \overline{C_n}}$, $b_n \in \overline{C_n}$ for every $n \in \mathbb{N}$ and the sequence $\{b_n\}$ converges to e, we have $e \in \overline{D} \subset \overline{SB} = SB$. By the property of κ-Fréchet-Urysohn of SB and D is open in SB, there is a sequence $L = \{c_k : k \in \mathbb{N}\} \subset D$ converging to e. On the other hand, since $e \notin \overline{C_n}$ for every $n \in \mathbb{N}$, the set $\{n \in \mathbb{N} : L \cap C_n \neq \varnothing\}$ is infinitely. Thus, we can put

$$\{n \in \mathbb{N} : L \cap C_n \neq \varnothing\} = \{n_i : i \in \mathbb{N}\}.$$

Hence, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $c_n \in C_{k_n}$. Moreover, it follows from

$$C_{k_n} \subset b_{k_n} B_{k_n} = p\left(b_{k_n}, q\left(a, A_{k_n}\right)\right)$$

that $c_n = p\left(b_{k_n}, q(a, x_n)\right)$ for some $x_n \in A_{k_n}$, for each $n \in \mathbb{N}$. Then, by Lemma 1.1, we have

$$q(a, x_n) = q\left(b_{k_n}, p\left(b_{k_n}, q(a, x_n)\right)\right) = q\left(b_{k_n}, c_n\right).$$

It is clear that $q(a, x_n) \to e$ by $b_n \to e$, $c_n \to e$ and the continuity of the mapping q. Lastly, since

$x_n = p\left(a, q(a, x_n)\right)$, it implies that $x_n \to a.$

Therefore, A is strongly κ-Fréchet-Urysohn.

Theorem 1.2. *The product of a κ-Fréchet-Urysohn rectifiable space G with a first-countable space M is strongly κ-Fréchet-Urysohn.*

Proof. Take any decreasing sequence $\{A_m : m \in \mathbb{N}\}$ of non-empty open sets in $G \times M$ and any point $(x, y) \in \bigcap_{m \in \mathbb{N}} \overline{A_m} \subset G \times M$. Then, since M is a first-countable space, we can choose $\{U_n : n \in \mathbb{N}\}$ as a decreasing countable neighborhood base at y in M such that U_n open for each $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, if we put

$$B_n = \pi_1\left[\left(G\times U_n\right)\cap A_n\right],$$

where $\pi_1 : G\times M \to G$ is the projection to the first coordinate, then $x\in\overline{B_n}$ for each $n\in\mathbb{N}$. In fact, since $(x,y)\in G\times U_n$, $G\times U_n$ is open and π_1 is continuous, by Lemma 1.11, we have

$$\begin{aligned} x=\pi_1(x,y)&\in\pi_1\left[\left(G\times U_n\right)\cap \overline{A_n}\right]\\ &\subset\pi_1\left[\overline{\left(G\times U_n\right)\cap \overline{A_n}}\right]\\ &=\pi_1\left[\overline{\left(G\times U_n\right)\cap A_n}\right]\\ &\subset\overline{\pi_1\left[\left(G\times U_n\right)\cap A_n\right]}=\overline{B_n}\end{aligned}$$

Moreover, it follows $U_{n+1}\subset U_n$ for each $n\in\mathbb{N}$ that $B_{n+1}\subset B_n$ for each $n\in\mathbb{N}$. On the other hand, since G, U_n and A_n are open, so B_n is open. Thus, $\{B_n\}$ is a decreasing sequence of open sets in G. Next, because G is strongly κ-Fréchet-Urysohn by Lemma 1.9, there exists a sequence $\{b_n : n\in\mathbb{N}\}$ such that $\{b_n : n\in\mathbb{N}\}$ converges to x and $b_n\in B_n$ for each $n\in\mathbb{N}$. For every $n\in\mathbb{N}$, choose $c_n\in U_n$ such that $(b_n,c_n)\in\left(G\times U_n\right)\cap A_n$.

Now, we will prove that $(b_n,c_n)\to(x,y)$. Indeed, let V be a neighborhood of y in M. Then, since $\{U_n : n\in\mathbb{N}\}$ is a countable neighborhood base at y in M, there exists $n_0\in\mathbb{N}$: $y\in U_{n_o}\subset V$. Moreover, since $\{U_n : n\in\mathbb{N}\}$ is a decreasing sequence, we have $c_n\in U_n\subset U_{n_o}\subset V$ for every $n\geq n_o$.

Thus, the sequence $\{c_n\}$ converges to y.

Lastly, since $b_n\to x$, $c_n\to y$, it implies that $(b_n,c_n)\to(x,y)$. Hence, $G\times M$ is strongly κ-Fréchet-Urysohn.

By means of Remark 1.8 and Theorem 1.2, we obtained the following corollary.

Corollary 1.3 ([8]). *The product of a κ-Fréchet-Urysohn rectifiable space G with a first-countable space M is κ-Fréchet-Urysohn.*

3.2. Evaluation

We give κ-Fréchet-Urysohn properties in rectifiable spaces and they are shown in Theorem 1.1, Theorem 1.2.

4.Conclusion

In this article, we give κ-Fréchet-Urysohn properties in rectifiable spaces. With these findings, we extend to a result in [8].

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TÍNH CHẤT κ – FRÉCHET-URYSOHN TRONG KHÔNG GIAN CẦU TRƯỜNG ĐƯỢC

Tóm tắt: Không gian tôpô G được gọi là không gian cầu trường được nếu tồn tại một phép đồng phôi $\varphi : G\times G\to G\times G$ và một phần tử $e\in G$ sao cho $\pi_1\circ\varphi=\pi_1$, và với mỗi $x\in G$ ta có $\varphi(x,x)=(x,e)$, trong đó $\pi_1 : G\times G\to G$ là phép chiếu lên tọa độ thứ nhất. Khi đó, phép đồng phôi φ được gọi là một phép cầu trường trên G và e gọi là phần tử đơn vị phải của G. Gần đây, không gian cầu trường được đã được nghiên cứu bởi nhiều tác giả và họ đã đặt ra nhiều câu hỏi mở mà đến nay vẫn chưa có lời giải đáp. Trong bài báo này, chúng tôi đưa ra các tính chất κ-Fréchet-Urysohn trong không gian cầu trường được. Nhờ những kết quả này, chúng tôi mở rộng một kết quả trong [8].

Từ khóa: Không gian cầu trường được; không gian κ-Fréchet-Urysohn; không gian κ-Fréchet-Urysohn mạnh; không gian thỏa mãn tiên đề đếm được thứ nhất; tập con compact.