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ON IDEMPOTENT - SEMIPRIME RINGS

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Abstract: A ring R is called idempotent-semiprime (briefly, idem-semiprime) if for any $a \in R, aea = 0$ for all idempotent $e \in I\left(R\right)$, implies $a = 0$, The class of idem-semiprime rings is a proper subclass of semiprime rings. This new class includes domains, reduced rings, and Von Neumann regular rings. In this article, we investigate the usual ring theoretic constructions of idempotent-semiprime rings.

Key words: idem-semiprime ring; semiprime ring; Von Neumann regular ring.

1. Introduction

In this note, we consider only nonzero (associative) rings with identity. The set of all idempotent elements of a ring R is denoted by $I(R)$.

In noncommutative rings, prime and semiprime rings are the important classes of rings. Their definitions are given as follows: an ideal P of a ring R is called prime if for any $a,b \in R, aRb \subseteq P$ implies $a \in P$ or $b \in P$, and semiprime if for any $a \in R, aRa \subseteq P$ implies $a \in P$. Further, a ring R is (semi)prime if is a (semi)prime ideal in R.

Our starting definition is obtained by formally replacing the whole ring in the above definitions, by , the set of all idempotents of . Thus, an ideal P of a ring R is *idempotent-prime (briefly, idem-prime)* if for any , for any idempotent of R, implies $a \in P$ or $b \in P$, and, *idempotent-semiprime (briefly, idem-semiprime)* if for any for any idempotent of R, implies .

Recall that an ideal P is completely prime if for any implies $a \in P$ or $b \in P$. Since , every completely prime ideal is idem-prime, that is, for ideals:

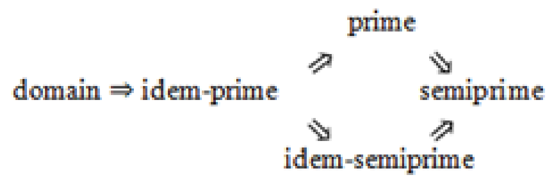
completely-prime idem-prime prime.

In a commutative ring, every prime ideal is completely prime and also is idem-prime.

A ring R is idem-(semi)prime if is an idem-(semi)prime ideal of R.

The main goal of this article is to investigate the class of idem-semiprime rings.

Reduced rings and Von Neumann regular rings are idem-semiprime. We have the follow chart



It is easy to see that in special cases, these new definitions coincide with the old ones. Example, a commutative ring is idem-prime iff it is prime iff it is an integral domain, and, is idem-semiprime iff it is semiprime iff it is reduced.

In Sec. 2, Ring theoretic constructions (i.e., product, quotients, polynomial rings) are studied, separating the matrix extensions in Section Three.

2. Ring theory constructions

By definition, a ring is an idem-semiprime iff for any for any , implies . By denial, a ring is an idem-semiprime ring iff for any , there is an idempotent with . Since for , the condition if obviously, the condition must be verified only for nonzero zero-square elements: is

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unit-semiprime iff for every with, there is an idempotent with.

Since intersection of idem-semprime ideals is an idem-semiprime ideal, we immediately obtain:

Proposition 1. *The set of all idem-semprime ideals of a ring, ordered by inclusion, form a complete lattice.*

Proof. For an arbitrary family of idem-semiprime ideals. The is the intersection . For an arbitrary subset of, consider is an idem-semiprime ideal in }. So and it is the smallest idem-semiprime ideal of R includes . So that the is

Proposition 2. *A product of rings is idem-semiprime iff each component is idem-semiprime.*

Proof. We only need to prove this for two rings . Suppose $R \times S$ is idem-semiprime anh let so that and . Since and is idem-semiprime, there are and such that .

Conversely, assume and . Further, suppose. Since is idem-semiprime and , there is an idempotent with. Thus,

Theorem 3. *Let be a ring homomorphism. Then*

(a) If is an idem-semiprime ideal of and then $f(P)$ *is idem-semiprime in .*

(b) If is an idem-semiprime ideal of and then is idem-semiprime in R *.*

Proof.

(a) For any, for any idempotent . There is . Since is an idempotent in , for any idempotent . Hence,

(b) For any, for any idempotent. Since, for any idempotent , there is an idempotent such that. So that, . Since P' is idem - semiprime.

Remark. For any ring homomorphism , we always have. To have the best possible correspondence between idem-semiprime ideals we need equality. For example, if denotes the canonical projection, the corresponding equality amount to lifting of idempotents modulo A .

Definition. Let be an ideal of ring . We say that *idempotents lift modulo* in case every idempotent , there is an idempotent such that.

Corollary 4. *Let be an ideal of a ring such that idempotents lift modulo . An ideal of the quotient ring is idem-semiprime iff it has the form with an idem-semiprime ideal of which includes.*

Proof. If is an ideal of the ring, where P is an idem-semiprime ideal of which includes, and is the canonical projection. Because , just apply Theorem 3(a) to , we have is an idem-semiprime ideal in . Conversely, if is an idem-semiprime ideal in , there is an ideal of such that. Because idempotents lift modulo I , , apply Theorem 3(b) to, we have is an idem-semiprime ideal in.

Corollary 5. Let be an ideal of a ring. If idempotents lift modulo then the factor ring is an idem-semiprime ring iff ideal is an idem-semiprime ideal of R .

Proof. From the Corollary 4.

Clearly, proving results on idem-semiprime polynomial rings depends on what extent we know the invertible polynomials. An easy example is: for any integral domain, the polynomial ring is idem-semiprime.

Because of commutative rings, idem-semiprime, semiprime and reduction are equivalent conditions, we obtain at once (see (10.18) in [4]).

Theorem 6. *Let be a set of variables which commute with one another as well as with elements of a ring . Then the polynomial ring is idem-prime (resp., idem-semiprime) iff* R *is idem-prime (resp., idem-semiprime).*

3. Matrix rings

Since the rings of triangular matrices are not even semiprime, the ring of triangular matrices over any ring is not idem-semiprime.

Theorem 7. *Matrix rings over idem-semiprime rings are idem-semiprime.*

Proof. The proof will be by induction on . The case being trivial. For , suppose that is idem-semiprime and take with is an row, is an column and , such that $A \neq 0$ and. We go into several cases.

Case 1.

Case 1.1. and. Here with nonzero column, assume. We use the row with all entries zero excepting the -th entry which we denote by and the idempotent matrix. Since we obtain

$$AEA = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta\gamma\beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta(t\beta_i) \\ 0 & 0 \end{bmatrix}$$

Therefore, if we can take and if, since is idem-semiprime, there is an idempotent such that , and we can take . In both case , as desired.

Case 1.2. and. Since, there is an idempotent with and for the idempotent matrix , we get

$$AEA = \begin{bmatrix} 0 & \beta \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} 0 & \beta \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \beta ed \\ 0 & ded \end{bmatrix} \neq 0_n.$$

Case 1.3. By induction hypothesis, there is an idempotent matrix such that . Then for the idempotent matrix , we obtain

$$AEA = \begin{bmatrix} M & \beta \\ 0 & d \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M & \beta \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} MVM & MV\beta \\ 0 & 0 \end{bmatrix} \neq 0_n.$$

Case 2. , say. Now we use the column with all entries zero excepting the -th entry denoted. Then, we take the idempotent matrix , we get

$$AEA = \begin{bmatrix} M & \beta \\ \alpha & d \end{bmatrix} \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M & \beta \\ \alpha & d \end{bmatrix}$$

$$= \begin{bmatrix} M\delta\alpha & M\delta\beta \\ \alpha\delta\alpha & \alpha\delta d \end{bmatrix}.$$

Therefore, if we can take and if , since is idem-semiprime, there is an idempotent such that , and we can take . In both case, so .

Proposition 8. *For any division ring* D *and positive integer , the matrix ring is idem-semiprime.*

Corollary 9. *A ring is semisimple iff it is idem-semiprime and left Artinian.*

Proof. If R is semisimple, by Wedderburn-Artin theorem, R is isomorphic to a product of finitely matrix rings over division rings. By proposition 2 and proposition 8, matrix rings over division rings is idem-semiprime. So R is idem-semiprime. The converse is (10.24) in [4].

Proposition 10. *Von Neumann regular rings are idem-semiprime.*

Proof. For any with. Because R R is regular, there exists an element with. It implies and. Choose, and compute

$$aea = a\left[xa + xax(1-xa)\right]a = axa^2$$
$$+axax(1-xa)a = ax\left(a - xa^2\right) = axa = a \neq 0.$$

4. Connections

In this section, we connect the idem-semiprime rings with some other well-known classes of rings.

Proposition 11. *A ring is domain if and only if* R *is idem-prime and idempotents commute with nilpotent elements.*

Proof. The conditions are clearly necessary. Conversely, assume that is an idem-prime ring and idempotents commute with nilpotent elements. Let with Then for all We have, so that bra commutes with idempotents. Let and for all Hence and so that, . Since is idem-prime, R also is prime, so for all By hypothesis, it implies or , and R is a domain.

Proposition 12. *A ring* R *is reduced if and only if* R *is idem-semiprime and idempotents commute with nilpotent elements.*

Proof. The conditions are clearly necessary. Conversely, assume is an idem-semiprime and idempotents commute with nilpotent elements. Let. Then a commute with idempotents and since R is idem-semiprime, for all idempotent It implies , and so R is reduced.

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VỀ VÀNH LUỸ ĐẲNG - NỬA NGUYÊN TỐ

Tóm tắt: Một vành R được gọi là luỹ đẳng-nửa nguyên tố nếu với mọi $a \in R, aea = 0$ với mọi luỹ đẳng e của R, thì suy ra $a = 0$. Lớp vành luỹ đẳng - nửa nguyên tố là một mở rộng thực sự của lớp vành nửa nguyên tố. Các vành gọn, miền, vành chính quy Von Neumann đều là các vành luỹ đẳng - nửa nguyên tố. Trong bài báo này, chúng tôi đưa ra một số kết quả của vành luỹ đẳng - nửa nguyên tố.

Từ khóa: vành luỹ đẳng - nửa nguyên tố; vành nửa nguyên tố; vành chính quy Von Neumann.