Tạp chí Khoa học Xã hội, Nhân văn và Giáo dục - ISSN: 1859 - 4603

UED JOURNAL OF SOCIAL SCIENCES, HUMANITIES & EDUCATION

Received:
20 – 09 – 2018
Accepted:
25 – 11 – 2018
http://jshe.ued.udn.vn/

SOME NEW RESULTS IN THE EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE VARIATIONAL INEQUALITY PROBLEM AND ITS APPLICATION

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Abstract: In this article, we consider the existence and uniqueness of the solution to the variational inequality problems and applying these results to investigate the convergence and the convergent rate of a projection method for solving the problem. At first, we introduce the variational inequality problem in a general setting and some basic definitions. Then, we present normal results about the existence and uniqueness of the solution to this problem. After that, we prove new outcomes about the existence and uniqueness of the variational inequality problem. Finally, new results are used to study the convergence and convergence rate of the projection method to the variational inequality problem.

Key words: variational inequality problem; existence of solution; uniqueness of solution; projection method; convergence; convergent rate.

1. Introduction

The systematic study of variational inequality problems began in the the mid-1960s and have become a wide range of important tools to investigate and solve abundant problems including equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Variational inequality theory is also closely related to optimization problems [2, 5].

The variational inequality problem is one of the central problems in nonlinear analysis. Hence, it attracts a huge number of investigation from numerous researchers in dynamic files in both theory and applications, such as the existence and uniqueness proposition of its solution, the algorithm to solve the solution and the quantities of its applications [6, 12,13,1,8-11]. A number of results about the existence and uniqueness of variational inequality problems were found in [2, 4]. Also, numerous algorithms to solve problems about variational inequality were introduced and the convergence and the rate of its convergence were proved. (See [3, 7,12,13] for details.)

In this article, we only concentrate on investigating the existence and uniqueness of variational inequality problems in the general form. Specially, we reintroduce some results and produce new outcomes about the existence and uniqueness of its solution. Some applications of new results in our article are related to the proof of convergence and rate of convergence of the projection method in variational equality problems in [2,6,12,13,1,8-11]. Given the application of our results, the unique assumptions of solutions in [2, Proposition 5.3] and [2, Proposition 5.4] are not necessary.

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2. Preliminaries

In this section, we introduce the variational inequality problem in the general form, present some basic definitions and some known results about the existence and uniqueness of the solution to the variational problem. These definitions and results are closely relaed to our new outcomes, which are presented in the next section.

Problem 2.1. (Variational inequality problem) *Given a nonempty, subset* C *of the Euclidean n-dimensional space* R^n *and a mapping* $F: C \to \mathrm{R}^n$, *the general variational inequality problem, denoted VI(F,C), is to find a vector* $x^* \in C$ *such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C.$$

The set of solutions to this problem is denoted $SOL(F,C)$.

Definition 2.1. Let C be nonempty, convex subset in R^n, and let $F: C \to \mathrm{R}^n$ be a mapping. The mapping F is said to be

(a) Strongly monotone on C with constant $\beta > 0$ if for all pair of points $x, y \in C$

$$\langle F(x) - F(y), x - y \rangle \geq \beta \| x - y \|^2 . \quad (1)$$

(b) Strictly monotone on C if for all distinct $x, y \in C$

$$\langle F(x) - F(y), x - y \rangle > 0. \quad (2)$$

(c) Monotone on C if for all pair of points $x, y \in C$, we have

$$\langle F(x) - F(y), x - y \rangle \geq 0. \quad (3)$$

(d) γ-Strongly pseudomonotone on C if for all pair of points $x, y \in C$, we have

$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq \gamma \| x - y \|^2 . \quad (4)$$

(e) Pseudomonotone on C if for all pair of points $x, y \in C$, we have

$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq 0. \quad (5)$$

(f) Quasimonotone on C if for all pair of points $x, y \in C$, we have

$$\langle F(y), x - y \rangle > 0 \Rightarrow \langle F(x), x - y \rangle \geq 0. \quad (6)$$

(g) Lipschitz continuous on C with $L > 0$ if for all $x, y \in C$, we have

$$\| F(x) - F(y) \| \leq L \| x - y \| . \quad (7)$$

It follows from the definitions that the following implications hold:

$$(a) \to (b) \to (c) \to (e) \to (f), (e) \leftarrow (d).$$

Definition 2.2. Given C be a nonempty set, convex subset of R^n. A mapping $F: C \to \mathrm{R}^n$ defined by

$$P_C(x) = argmin\{\| x - y \| : y \in C\},$$

is called projector onto C.

Proposition 2.1 ([2, Proposition 1.2]) *Let* C *be a closed convex set. Then* $y = P_C(x)$ *iff*

$$\langle y - x, z - y \rangle \geq 0, \forall z \in C. \quad (8)$$

In variational inequality $VI(F,C)$ for each $x \in C$ and $\lambda > 0$, mapping $F_C: C \to \mathrm{R}^n$ defined by

$$F_C(x) = x - P_C(x - \lambda F(x)), \quad (9)$$

is often called *natural map* of F onto C. The relation between the solution of variational inequality $VI(F,C)$ and the natural map is showed in the following results.

Proposition 2.2 ([1, Proposition 2.2]). x^* *is a solution of variational inequality problem* $VI(F,C)$ *if* x^* *is a zero-point of mapping* F_C, *i.e* $F_C(x^*) = 0$.

Based on Proposition 2.2 and Brouwer fixed-point theory (see [2] for details), the results about the existence and uniqueness of solution in variational equality have been proven. For the proof, we refer the reader to [2, 4].

Proposition 2.3 ([1, Proposition 2.1]). *Let* C *be a closed, compact and convex subset of* R^n *and a continuous mapping* $F: C \to \mathrm{R}^n$. *The set* $SOL(F,C)$ *is nonempty.*

Proposition 2.4 ([2, Corollary 2.1]). *Let* C *be a nonempty, closed and convex subset of* R^n *and let* $F: C \to \mathrm{R}^n$ *be a continuous. If* F *is coercive mapping, i.e, exist* $x^0 \in C$ *such that*

$$\frac{\langle F(x) - F(x^o), x - x^o \rangle}{\| x - x^o \|} \to +\infty, \quad \text{as} \quad \| x \| \to +\infty \quad \text{for}$$

$x \in C$ and for some $x^0 \in C$, then variational inequality $VI(F,C)$ has a solution.

Proposition 2.5 ([4, Propostion 1.4]) *If* C *is a nonempty, closed and convex subset of* R^n *and*

$F : C \to R^n$ be β-strongly pseudomonotone and Lipschitz continuous on C then variational inequality $VI(F,C)$ has unique solution.

3. Main results

Theorem 3.1. *If F is a strictly monotone on C, then the problem $VI(F,C)$ has at most one solution.*

Proof. Assume that x_1 and x_2 are solutions to Problem $VI(F,C)$, $x_1 \neq x_2$. Then we have

$$\langle F(x_1), x - x_1\rangle \geq 0, \forall x \in C \quad (10)$$

and

$$\langle F(x_2), x - x_2\rangle \geq 0, \forall x \in C. \quad (11)$$

Replace x to x_1 in (11) and x to x_2 in (10), then add these two inequalities to yield:

$$\langle F(x_1) - F(x_2), x_2 - x_1\rangle \geq 0, \forall x \in C$$

or

$$\langle F(x_1) - F(x_2), x_1 - x_2\rangle \leq 0, \forall x \in C.$$

This contradict with the definition of strictly monotone function F. Hence $x_1 = x_2$.

Corollary 3.1. *Let C be a nonempty, compact and convex subset of R^n, and let $F : C \to R^n$ be a continuous and strictly monotone mapping on C. Then variational inequality $VI(F,C)$ has a unique solution.*

Proof. From the Proposition 2.3 we obtain the existence of solution in problem $VI(F,C)$. The uniqueness of solution is directly deduced from the Theorem 3.1.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of R^n and let $F : C \to R^n$ be a continuous. If F is a continuous, strictly monotone and coercive mapping, i.e, there exists $x^0 \in C$ such that*

$$\frac{\langle F(x) - F(x^o), x - x^o\rangle}{\| x - x^o \|} \to +\infty,$$

as $\|x\| \to +\infty$ for $x \in C$ and for some $x^0 \in C$, *then variational inequality $VI(F,C)$ has a unique solution.*

Proof. From the Proposition 2.4 we have the existence of solution in problem $VI(F,C)$, the uniqueness of solution is deduced from Theorem 3.1.

Theorem 3.2. *If F be γ-strongly pseudomonotone on C with $\gamma > 0$ then the $VI(F,C)$ has a unique solution.*

Proof. Assume x_1 and x_2 are two solutions to the problem $VI(C,F)$ and $x_1 \neq x_2$. Then, we have

$$F(x_1), x_2 - x_1\rangle \geq 0, \forall x \in C \quad (12)$$

and

$$\langle F(x_2), x_1 - x_2\rangle \geq 0, \forall x \in C. \quad (13)$$

From (12) and the proposition of γ-strongly pseudomonotone of F on C we deduce that

$$\langle F(x_2), x_2 - x_1\rangle \geq \gamma \| x_2 - x_1 \|^2 > 0 \quad (14)$$

We can rewrite this inequality as

$$\Rightarrow \langle F(x_2), x_1 - x_2\rangle < 0. \quad (15)$$

This contrasts with (13). Hence, $x_1 = x_2$, or the problem $VI(F,C)$ has a unique solution.

Corollary 3.3. *Let C be a nonempty, compact and convex subset of R^n and let $F : C \to R^n$ be a continuous mapping and γ-strongly pseudomonotone on C with $\gamma > 0$. Then, the $VI(F,C)$ has a unique solution.*

Proof. From the Proposition 2.3, we deduce the existence of the solution in the $VI(F,C)$. The uniqueness of the solution comes from the Theorem 3.2.

Corollary 3.4. *Let C be a nonempty, compact and convex subset of R^n and let $F : C \to R^n$ be a continuous mapping and γ-strongly pseudomonotone on C with $\gamma > 0$ and satisfies the coercivity condition. Then, the $VI(F,C)$ has a unique solution.*

Proof. From the Proposition 2.5, we can obtain the existence of the solution in the problem $VI(F,C)$ and the uniqueness of its solution can be obtained from Theorem 3.2.

4. Application

In this section, we will use new results in previous parts in order to receive the convergence and the rate of convergence of a projection method for the $VI(F,C)$. At first, we consider the projection method for the $VI(F,C)$ that was presented in [1].

Algorithm 1:

Step 1. Compute $x^{k+1} := P_C(x^k - \lambda_k F(x^k))$. If $x^{k+1} = x^k$ then stop. Otherwise, return to Step 2.

Step 2. Set $k := k+1$ and return to Step 1.

If Algorithm stops in step k, then x^k is a solution to the $VI(F,C)$, see [2]. Otherwise, if Algorithm has infinite iteration, then the convergence and the rate of convergence are proven in the following theorem.

Theorem 4.1 ([2, Theorem 5.3]). *Let C be a non empty, closed, convex subset of $\mathbb{R}^n$ and let $F : C \to \mathbb{R}^n$ be Lipschitz continuous with a constant $L > 0$ and γ-strongly pseudomonotone with $\gamma > 0$. Choose iterated consequence λ_k such that*

$$0 < a < \lambda_k \le b < \frac{2\gamma}{L^2}, \forall k \ge 0, \tag{16}$$

with a,b be positive numbers. Let x^k be defined by Algorithm 1. If x^* be a unique solution of $VI(F,C)$ then the consequence $\{x^k\}$ converges linear to x^*. Furthermore, the rate of convergence is defined by

$$\|x^{k+1} - x^*\| \le \frac{\mu^{k+1}}{1-\mu}\|x^0 - x^1\| \tag{17}$$

and

$$\|x^{k+1} - x^*\| \le \frac{\mu}{1-\mu}\|x^{k+1} - x^k\|, \forall k \ge 0, \tag{18}$$

as

$$\mu := \frac{1}{\sqrt{1 + a(2\gamma - bL^2)}} \in (0;1). \tag{19}$$

Theorem 4.2 ([2, Theorem 5.4]). *Let C be a nonempty, closed, convex subset of $\mathbb{R}^n$ and $F : C \to \mathbb{R}^n$ be Lipschitz continuous with a constant $L > 0$ and γ-strongly pseudomonotone with $\gamma > 0$. Choose iterated consequence λ_k such that*

$$\lim_{h \to 0} \lambda_k = 0, \sum_{k=0}^{\infty} \lambda_k = +\infty. \tag{20}$$

Assume that the $VI(F,C)$ has a unique solution x^*, then consequence x^k is defined by Algorithm 1 converges to x^*. Moreover, there exists k_0 such that

$$\lambda_k(2\gamma - \lambda_k L^2) > 0, \tag{21}$$

$$\|x^{k+1} - x^*\| \le \frac{1}{\sqrt{\prod_{i=k_0}^{k}[1 + \lambda_i(2\gamma - \lambda_i L^2)]}}\|x^{k_0} - x^*\|, \tag{22}$$

for all $k \ge k_0$.

One of the conditions for obtaining results about the convergence and the rate of convergence of Algorithm 1 that is received in the Theorem 4.1 and 4.2 is the uniqueness of the solution in the $VI(F,C)$. Combining it with Corollary 3.3 and 3.4, we obtain the following results:

Corollary 4.1. *Let C be a nonempty, closed, compact and convex subset of $\mathbb{R}^n$ and let $F : C \to \mathbb{R}^n$ be Lipschitz continuous with a constant $L > 0$ and γ-strongly pseudomonotone with $\gamma > 0$. Choose iterated consequence λ_k such that*

$$0 < a < \lambda \le b < \frac{2\gamma}{L^2}, \forall k \ge 0, \tag{23}$$

with a,b be positive numbers. Let x^k be defined by Algorithm 1. Then the consequence $\{x^k\}$ converges linear to x^*. Furthermore, the rate of convergence is defined by

$$\|x^{k+1} - x^*\| \le \frac{\mu^{k+1}}{1-\mu}\|x^0 - x^1\| \tag{24}$$

and

$$\|x^{k+1} - x^*\| \le \frac{\mu}{1-\mu}\|x^{k+1} - x^k\|, \forall k \ge 0, \tag{25}$$

where

$$\mu := \frac{1}{\sqrt{1+a(2\gamma - bL^2)}} \in (0;1). \qquad (26)$$

Proof. From the Theorem 3.2, we obtain the unique solution x^*. The rest of corollary directly comes from Theorem 4.1.

Corollary 4.2. *Let C be a nonempty, closed, compact and convex subset of $\mathbb{R}^n$ and let $F : C \to \mathbb{R}^n$ be Lipschitz continuous with a constant $L > 0$ and satisfies the coercivity condition. Choose positively iterated consequence λ_k such that*

$$\lim_{h\to 0}\lambda_k = 0, \sum_{k=0}^{\infty}\lambda_k = +\infty. \qquad (27)$$

Then the consequence $\{x^k\}$ converges to x^*. Furthermore, there exists k_0 such that

$$\lambda_k(2\gamma - \lambda_k L^2) > 0, \qquad (28)$$

$$\|x^{k+1} - x^*\| \le \frac{1}{\sqrt{\prod_{i=k_0}^{k}[1+\lambda_i(2\gamma-\lambda_i L^2)]}}\|x^{k_0} - x^*\| \qquad (29)$$

for all $k \ge k_0$.

Proof. From the Theorem 3.2, we deduce the uniqueness of the solution. The rest of corollary is obtained from Theorem 4.2.

5. Conclusion

In this article, we prove two new results about the uniqueness of variational inequality problems $VI(F,C)$, Theorem 3.1 and 3.2. Basing on these results and combining with knowledge of the existence of the solution, we gain four new outcomes about the existence and the uniqueness of the solution in the $VI(F,C)$, Corollary in section 3.

Some new results in this article are of great significance. Their roles are illustrated through the application in examining the convergence and the rate of convergence of the projection method (Algorithm 1). We also ignore the assumption about the uniqueness of the $VI(F,C)$ by assuming an additional condition either onto C (C has compactness) or function F (F has coercivity condition). In this case, we receive two particular cases to obtain the convergence and the rate of convergence of Algorithm 1 for the $VI(F,C)$. This contribute to the clarification of known results, Theorem 4.1 and 4.2.

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MỘT SỐ KẾT QUẢ MỚI VỀ SỰ TỒN TẠI VÀ DUY NHẤT NGHIỆM CỦA BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN VÀ ỨNG DỤNG

Tóm tắt: Trong bài báo này, chúng tôi xét sự tồn tại và tính duy nhất nghiệm của bài toán bất đẳng thức biến phân và ứng dụng để nghiên cứu sự hội tụ và tốc độ hội tụ của một phương pháp chiếu để giải bài toán này. Trước hết, chúng tôi phát biểu bài toán bất đẳng thức biến phân và một số khái niệm liên quan. Sau đó, chúng tôi trình bày các kết quả đã biết về sự tồn tại, duy nhất nghiệm của bài toán. Tiếp đến, chúng tôi trình bày các kết quả mới đạt được về tính duy nhất nghiệm và sự tồn tại duy nhất nghiệm của bài toán bất đẳng thức biến phân. Cuối cùng, các kết quả mới được sử dụng vào việc nghiên cứu sự hội tụ và tốc độ hội tụ của phương pháp chiếu một lần cho bài toán bất đẳng thức biến phân.

Từ khóa: bài toán bất đẳng thức biến phân; sự tồn tại nghiệm; tính duy nhất nghiệm; phương pháp chiếu; sự hội tụ; tốc độ hội tụ.